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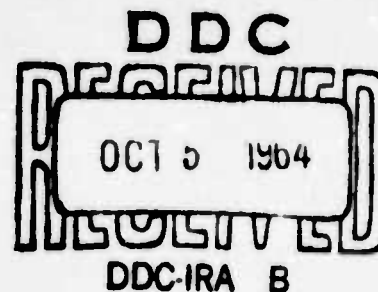
MEMORANDUM
RM-4277-PR
SEPTEMBER 1964

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THE APPLICATION OF THE KALMAN FILTER TO SOME ASTRODYNAMICAL PROBLEMS

F. T. Smith



PREPARED FOR:
UNITED STATES AIR FORCE PROJECT RAND

The **RAND** Corporation
SANTA MONICA • CALIFORNIA

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PREFACE

This Memorandum is a result of RAND's continuing study of guidance and orbit mechanics. It was written to elaborate on the analytical details of the computational equations for a Kalman filter, discussed in RM-4241-PR, The Application of Dynamic Programming to Satellite Intercept and Rendezvous Problems, F. T. Smith, August 1964. Such a filter was used to process observational data and to estimate relative state variables for the optimal control process.

In addition, this Memorandum discusses the application of the Kalman filter to orbit determination and orbit correction processes when these processes are described in terms of two-body orbital parameters. It may be of use in studies of the application of dynamic programming to flight control problems.

SUMMARY

This Memorandum applies dynamic programming to the derivation of the computational equations for a Kalman filter.

The problem of using a set of ill-conditioned observations is considered. For example, when observations of a satellite taken from a single ground station are spaced too close together in time, the accuracy of the determination of the satellite's orbital parameters is poor. Such a set of observations is said to be ill-conditioned.

The general set of computational equations derived is interpreted in terms of an orbit determination problem, an orbit correction process, and a satellite rendezvous problem.

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LIST OF SYMBOLS

A_i	azimuth angle at time t_i
A_k	6 x 3 matrix associated with optimal control process
$A(t)$	6 x 6 matrix in differential equation for relative system state vector
A_τ	6 x 6 constant matrix associated with target orbit
a	orbital semi-major axis
B	6 x 3 constant matrix in differential equation for relative system state vector
C	m x 6 matrix associated with differential correction process
E_o	eccentric anomaly at time t_o
e	orbital eccentricity
$f_m[z(m)]$	cost of m-stage observation process
$g(t, t_k)$	change in relative system state vector associated with target motion
H	matrix associated with least squares method
h_i	elevation angle at time t_i
J	performance index associated with Kalman filter
$M(t)$	Jacobian matrix for transformation equations from observation variables to relative system state vectors
P_m	covariance matrix
$p(t)$	orbital parameter vector
Q_k	matrix associated with optimal control process
R	covariance matrix
$s(t)$	system state vector
$\vec{U}_o, \vec{V}_o, \vec{W}$	set of orthogonal unit vectors
$u(t)$	optimal control vector

v	vector whose components are random variables
x	parameter vector
y	true observation vector
z	actual observation vector
$\Gamma(\tau-t)$	3 x 6 matrix determining optimal control vector
Δf_m	change in cost function due to utilization of a new observation
Δp	orbital parameter correction vector
Δq	true observation vector
$\Delta s(t)$	relative system state vector
Δv_k	incremental velocity vector associated with optimal control process
$\Delta \tilde{u}_o, \Delta \tilde{v}_o, \Delta \tilde{w}_o$	small rotations about $\tilde{U}_o, \tilde{V}_o, \tilde{W}$ respectively
ρ	range from observing site to vehicle
σ	square root of variance

I. INTRODUCTION

This Memorandum uses the techniques of dynamic programming to formulate the Kalman filtering problem and to obtain the necessary computational equations. The discussion is less general than that in Ref. 1, and the point of view here is slanted more toward astrodynamical applications. The application of dynamic programming to this type of problem is discussed in Refs. 2 and 3. This Memorandum goes into much more detail concerning applications than do Refs. 2 and 3.

The method of least squares is briefly discussed to establish notation. The problem is then formulated as a multi-stage decision process, and dynamic programming concepts are applied to obtain the necessary computational equations. A method of treating ill-conditioned systems⁽⁴⁾ is incorporated into the computational equations. The application of the estimation process to the determination of a set of two-body orbital parameters from a sequence of observations is then discussed, i.e., the orbit determination problem. The discussion is then extended to feedback control problems where it is desired to estimate time-varying corrections to two-body orbital parameters and system state vectors.

II. THE METHOD OF LEAST SQUARES

Consider a set of m observations of one or more physical quantities which have been corrupted by noise. If y_i is the true value of each quantity at the time of the i -th observation, and v_i is the error due to noise, then the observed value z_i is

$$z_i = v_i + y_i, \quad i = 1, \dots, m$$

Assume that the y_i are related to a set of n parameters x_j whose values are to be determined from the relation

$$y = Hx$$

where*

$$y_i = m \times 1 \text{ vector } [y_1, \dots, y_m]^T$$

$$x_j = n \times 1 \text{ vector } [x_1, \dots, x_n]^T$$

$H = m \times n$ matrix whose elements may be
time varying

Since the true values of the observed quantities are corrupted by noise, the best we can hope to do is to solve the equation

$$z = Hx + v$$

This equation may be written as

$$v = z - Hx$$

where

$$v = [v_1, \dots, v_m]^T$$

*The superscript T denotes the transpose of a vector or matrix.

We shall assume that vector v represents a gaussian random process with zero mean. According to the principle of least squares the best estimate of vector x will be that for which the quadratic form

$$J = v^T R^{-1} v$$

is a minimum, where R is the covariance matrix of the gaussian process. The well-known solution to this problem is

$$\hat{x} = (H^T R^{-1} H)^{-1} H^T R^{-1} z$$

If this solution is substituted into the quadratic form representing J we obtain

$$\begin{aligned} \text{Min } J &= \left[z - H(H^T R^{-1} H)^{-1} H^T R^{-1} z \right]^T R^{-1} \left[z - H(H^T R^{-1} H)^{-1} H^T R^{-1} z \right] \\ &= z^T \left[R^{-1} - R^{-1} H(H^T R^{-1} H)^{-1} H^T R^{-1} \right] z = z^T Q(m) z \end{aligned}$$

If matrix H is square and non-singular, $\text{Min } J$ is zero.

Suppose that we now acquire a new observation z_{m+1} , and we wish to use z_{m+1} to improve further our estimate of x . We can form an $m+1$ system of equations and solve as above to obtain \hat{x} where H is now an $(m+1) \times n$ matrix.

As we combine more and more observations the estimate \hat{x} approaches closer to the true vector x , but the computational process becomes more and more inefficient since the dimensions of H and R grow with each observation. It is desirable, therefore, to find a more efficient method of utilizing new observations.

III. THE APPLICATION OF DYNAMIC PROGRAMMING

The process of estimating the components of vector x becomes a multi-stage decision process when new observations are combined sequentially to obtain successively improved estimates of x . Further, each new decision requires a new minimization operation.

We shall call the vector^{*}

$$z(m) = [z_1, \dots, z_m]^T$$

the observation vector. At each stage of the process vector z increases its dimension by one due to the addition of a new observation. The earlier components of z , of course, remain unchanged. At each stage of the process the decision involves choosing \hat{x} such that^{**}

$$J_m = v^T(m) R^{-1}(m) v(m)$$

where

$$v(m) = [v_1, \dots, v_m]^T$$

is minimized where m is the number of the observations involved.

It is clear that the estimate of x depends on the number and the values of the components of $z(m)$, as will the value of J_m . Instead

^{*}To clarify notation regarding the dimension of vectors z and v , we shall let $z(m)$ and $v(m)$ denote the fact that z and v are m -dimensional vectors. Vector x is assumed to be always of dimension n .

^{**}The notation $R(m)$ implies an $m \times m$ matrix.

of referring to an N-stage decision process we shall refer to an m-observation estimation process and define the minimum value of J_m as

$$f_m[z(m)] = \text{The cost of an m-observation estimation process based on an m-dimensional observation vector } z(m) \text{ with an optimal estimation policy being used.}$$

By an optimum estimation policy is meant the choice of successive estimates of x such that the performance index

$$J_m = v(m)^T R^{-1}(m) v(m) \quad m = n, n+1, \dots$$

is minimized. The reason for index m starting at $m = n$ is that a minimum of n observations are required to obtain a useful estimate of the n components of vector x . It is assumed that the components of $v(m)$ are independent, i.e., $R^{-1}(m)$ is a $m \times m$ diagonal matrix.

Let a new observation z_{m+1} be added as the $(m+1)$ -th component to vector $z(m)$. A new component v_{m+1} , the uncertainty associated with observation z_{m+1} , is also added to vector $v(m)$. If R_{m+1} is the variance associated with v_{m+1} , then we can write J_{m+1} as*

$$\begin{aligned} J_{m+1} &= v_{m+1}^T R_{m+1}^{-1} v_{m+1} + J_m \\ &= v_{m+1}^T R_{m+1}^{-1} v_{m+1} + v(m)^T R^{-1}(m) v(m) \end{aligned}$$

* Since v_{m+1} and R_{m+1}^{-1} are scalars $v_{m+1}^T R_{m+1}^{-1} v_{m+1}$ equals $v_{m+1}^2 / \sigma_{m+1}^2$. The vector-matrix notation is retained for convenience since it is possible to add several observations at a time to improve the estimate, and R_{m+1}^{-1} would then be a matrix.

The estimate of x that minimizes J_{m+1} , denoted by $(\hat{x})_{m-n+1}$, will be different from that minimizing J_m .^{*} Let the new estimate minimizing J_{m+1} be expressed as $(\hat{x})_{m-n} + (\Delta\hat{x})_{m-n}$. When the components of vector $(\hat{x})_{m-n}$ are changed to $(\hat{x})_{m-n} + \Delta x$ a new value for $J_m > f_m[z(m)]$ is obtained

$$J_m = \left\{ z(m) - H(m) \left[(\hat{x})_{m-n} + \Delta x \right] \right\}^T R^{-1}(m) \left\{ z(m) - H(m) \left[(\hat{x})_{m-n} + \Delta x \right] \right\}$$

Expanding this quadratic form and properly grouping terms yields

$$\begin{aligned} J_m = & z^T(m) R^{-1}(m) z(m) - 2 z^T(m) R^{-1}(m) H(m) (\hat{x})_{m-n} \\ & + (\hat{x})_{m-n}^T H^T(m) R^{-1}(m) H(m) (\hat{x})_{m-n} \\ & + 2 \Delta x^T \left[H^T(m) R^{-1}(m) H(m) (\hat{x})_{m-n} - H^T(m) R^{-1}(m) z^T(m) \right] \\ & + \Delta x^T H^T(m) R^{-1}(m) H(m) \Delta x \end{aligned}$$

The first three terms clearly are $f_m[z(m)]$ and the bracketed term vanishes from the definition of $(\hat{x})_{m-n}$ (See Section II). This leaves

$$J_m = \Delta x^T H^T(m) R^{-1}(m) H(m) \Delta x + f_m[z(m)]$$

^{*}The notation $(\hat{x})_{m-n}$ denotes the $(m-n)$ -th estimate of vector x and is used to distinguish the $(m-n)$ -th estimate from the $(m-n)$ -th component denoted by x_{m-n} .

and J_{m+1} becomes*

$$J_{m+1} = \left\{ z_{m+1} - H_{m+1} \left[(\hat{x})_{m-n} + \Delta x \right] \right\}^T R_{m+1}^{-1} \left\{ z_{m+1} - H_{m+1} \left[(\hat{x})_{m-n} + \Delta x \right] \right\} \\ + \Delta x^T H^T(m) R^{-1}(m) H(m) \Delta x + f_m \left[z(m) \right]$$

We can now apply the principle of optimality to derive a recurrence relation for $f_m \left[z(m) \right]$. For this problem the principle of optimality can be stated as follows:

An optimal sequence of estimations of the parameter vector, $(\hat{x})_1, (\hat{x})_2, \dots$, has the property that whatever choice is made for $(\hat{x})_1$, the remaining sequence of estimates $(\hat{x})_2, (\hat{x})_3, \dots$, must constitute an optimal sequence with regard to the observations from which $(\hat{x})_1$ is estimated.

We obtain the following recurrence relations:

$$f_{m+1} \left[z(m+1) \right] = \min_{\Delta x} \left\{ \left\{ z_{m+1} - H_{m+1} \left[(\hat{x})_{m-n} + \Delta x \right] \right\}^T R_{m+1}^{-1} \left\{ z_{m+1} - H_{m+1} \left[(\hat{x})_{m-n} + \Delta x \right] \right\} \right. \\ \left. + \Delta x^T H^T(m) R^{-1}(m) H(m) \Delta x + f_m \left[z(m) \right] \right\}$$

*The symbol H_{m+1} denotes the $n \times 1$ vector associated with observation z_{m+1} . If z_{m+1} denotes several simultaneous observations H_{m+1} is a matrix.

where

$$f_n [z(n)] = z^T(n) Q(n) z(n)$$

$$m = n, n+1, \dots$$

The first term inside the braces represents the direct cost of using the $(m+1)$ -th observation. The second cost represents the change in the cost for the previous m observations due to changing the estimate of x .

Noting that $f_m [z(m)]$ is independent of Δx , we minimize the terms inside the braces by differentiating with respect to the components of Δx . Setting the result equal to zero and solving for $(\Delta \hat{x})_{m-n}$ yields

$$(\Delta \hat{x})_{m-n} = \left[H^T(m) R^{-1}(m) H(m) + H_{m+1}^T R_{m+1}^{-1} H_{m+1} \right]^{-1} H_{m+1}^T R_{m+1}^{-1} \left[z_{m+1} - H_{m+1} (\hat{x})_{m-n} \right]$$

The new estimate for x is then

$$(\hat{x})_{m-n+1} = (\hat{x})_{m-n} + (\Delta \hat{x})_{m-n}$$

It is clear that J_{m+1} can be expressed as

$$J_{m+1} = \left[v(m) \mid v_{m+1} \right]^T \begin{bmatrix} R^{-1}(m) & 0 \\ 0 & R_{m+1}^{-1} \end{bmatrix} \begin{bmatrix} v(m) \\ v_{m+1} \end{bmatrix}$$

$$= v^T(m+1) R^{-1}(m+1) v(m+1)$$

where $R^{-1}(m+1)$ denotes the partitioned matrix.

Since

$$\begin{bmatrix} v(m) \\ \text{---} \\ v_{m+1} \end{bmatrix} = \begin{bmatrix} z(m) \\ \text{---} \\ z_{m+1} \end{bmatrix} - \begin{bmatrix} H(m) \\ \text{---} \\ H_{m+1} \end{bmatrix} \begin{bmatrix} x \end{bmatrix}$$

it is clear from Section II that $f_{m+1}[z(m+1)]$ must be given by

$$f_{m+1}[z(m+1)] = z^T(m+1) Q(m+1) z(m+1)$$

where

$$Q(m+1) = R^{-1}(m+1)$$

$$- R^{-1}(m+1) H(m+1) \left[H^T(m+1) R^{-1}(m+1) H(m+1) \right]^{-1} H^T(m+1) R^{-1}(m+1)$$

$$H(m+1) = \begin{bmatrix} H(m) \\ \text{---} \\ H_{m+1} \end{bmatrix}$$

It follows from the nature of J_m as expressed by the equation

$$J_m = \Delta x^T H^T(m) R^{-1}(m) H(m) \Delta x + f_m[z(m)]$$

that each correction to the estimate of vector x for each stage of the decision process can be computed by the equation

$$\begin{aligned} (\Delta \hat{x})_{m-n} = & \left[H^T(m) R^{-1}(m) H(m) + H_{m+1}^T R_{m+1}^{-1} H_{m+1} \right]^{-1} H_{m+1}^T R_{m+1}^{-1} \begin{bmatrix} z_{m+1} \\ \text{---} \\ z_{m+1} \end{bmatrix} \\ & - H_{m+1} (\hat{x})_{m-n} \end{aligned}$$

where

$$(\hat{x})_{m-n+1} = (\hat{x})_{m-n} + (\Delta\hat{x})_{m-n}$$

$$m = n, n+1, \dots$$

To avoid a matrix inversion process we follow the notation of Ref. 1 and define

$$P_m^{-1} = H^T(m) R^{-1}(m) H(m)$$

$$P_{m+1}^{-1} = P_m^{-1} + H_{m+1}^T R_{m+1}^{-1} H_{m+1}$$

Then by the lemma given in Ref. 1 we have*

$$P_{m+1} = P_m - P_m H_{m+1}^T (H_{m+1} P_m H_{m+1}^T + R_{m+1})^{-1} H_{m+1} P_m$$

If we now replace our index notation by $k = m-n$ we have the following set of recurrence relations:

$$(\hat{x})_{k+1} = (\hat{x})_k + (\Delta\hat{x})_k$$

$$(\Delta\hat{x})_k = P_{k+1} H_{k+1}^T R_{k+1}^{-1} [z_{k+1} - H_{k+1} (\hat{x})_k]$$

$$P_{k+1} = P_k - P_k H_{k+1}^T (H_{k+1} P_k H_{k+1}^T + R_{k+1})^{-1} H_{k+1} P_k$$

$$k = 0, 1, 2, \dots$$

*This can be proved by direct substitution of the expressions for P_{m+1}^{-1} and P_{m+1} into $P_{m+1}^{-1} P_{m+1} = I$.

In order to use these recurrence relations it is necessary to have a set of initial conditions $(\hat{x})_0$ and P_0 as well as R_1, R_2, \dots .

The quantity z_{k+1} can represent a single observation or a vector whose components represent several observations. The computational advantage of utilizing observations one at a time is that $(H_{k+1} P_k H_{k+1}^T + R_{k+1})$ and R_{k+1} are scalars and matrix inversion is eliminated entirely.

The selection of suitable initial conditions will be discussed in the sections to follow.

IV. ILL-CONDITIONED SYSTEMS

Consider the equation relating the true observation vector y and the parameter vector x given in Section II.

$$y = H x$$

In order that this equation have a unique solution the rank of matrix H must equal the dimension of vector x . Equivalently, we may require that the determinant of matrix $H^T H$, i.e., the Gram determinant of H , be non-zero. While this condition satisfies the theoretical requirement for a unique solution, it may not satisfy computational requirements for finding such a solution.

When the Gram determinant has a sufficiently small numerical value, the computation of the components of vector x is subject to considerable error and the set of observations defining vector y is said to be ill-conditioned.

The recurrence equations given in the last section may converge too slowly, or not at all, if they are applied to an ill-conditioned set of observations.

A method for treating ill-conditioned linear systems is given by Bellman, Kalaba, and Lockett.⁽⁴⁾ The recurrence relations given in Section III can be modified to incorporate one of these computational techniques for ill-conditioned systems.

Consider first the one-stage process in Section II. We modify the performance index to be

$$J_m = \left[z(m) - H(m)x \right]^T R^{-1}(m) \left[z(m) - H(m)x \right] + \lambda(x-c)^T (x-c)$$

where c is an $n \times 1$ vector chosen so that the norm $\|x-c\|$ is small.

If we expand J_m , differentiate with respect to the components of vector x , set the result equal to zero, and solve for \hat{x} , we obtain

$$\hat{x} = \left[H^T(m) R^{-1}(m) H(m) + \lambda I \right]^{-1} H^T(m) R^{-1}(m) z(m) + \lambda \left[H^T(m) R^{-1}(m) H(m) + \lambda I \right]^{-1} c$$

as the best least squares estimate. Reference 4 describes a method of successive approximations where vector c is taken as $(\hat{x})_1$ and \hat{x} becomes $(\hat{x})_2$. We then have the recurrence equation

$$(\hat{x})_{k+1} = \left[H^T(m) R^{-1}(m) H(m) + \lambda I \right]^{-1} \left\{ H^T(m) R^{-1}(m) z(m) + \lambda (\hat{x})_k \right\}$$

which, as shown in Ref. 4, converges to x .

Assume that a new observation z_{m+1} becomes available whose uncertainty is v_{m+1} . We know that the optimal estimate of x will change. The value of J_m for $(\hat{x})_{m-n} + \Delta x$ becomes

$$J_m = \left\{ z(m) - H(m) \left[(\hat{x})_{m-n} + \Delta x \right] \right\}^T R^{-1}(m) \left\{ z(m) - H(m) \left[(\hat{x})_{m-n} + \Delta x \right] \right\} + \lambda \left[(\hat{x})_{m-n} + \Delta x - c \right]^T \left[(\hat{x})_{m-n} + \Delta x - c \right]$$

If the subscripts and arguments are temporarily dropped for

notational convenience we can expand J_m as

$$\begin{aligned}
 J_m &= \left[z - H(\hat{x} + \Delta x) \right]^T R^{-1} \left[z - H(\hat{x} + \Delta x) \right] + \lambda (\hat{x} + \Delta x - c)^T (\hat{x} + \Delta x - c) \\
 &= z^T R^{-1} z - 2(\hat{x} + \Delta x)^T H^T R^{-1} z + (\hat{x} + \Delta x)^T (H^T R^{-1} H + \lambda I) (\hat{x} + \Delta x) \\
 &\quad - 2\lambda (\hat{x} + \Delta x)^T c + \lambda c^T c
 \end{aligned}$$

We next expand the terms involving $\hat{x} + \Delta x$ and regroup the terms to take advantage of

$$\left[H^T(m) R^{-1}(m) H(m) + \lambda I \right] (\hat{x})_{m-n} - H^T(m) R^{-1}(m) z(m) - \lambda c = 0$$

We obtain

$$\begin{aligned}
 J_m &= z^T R^{-1} z - 2\hat{x}^T H^T R^{-1} z + \hat{x}^T (H^T R^{-1} H + \lambda I) \hat{x} - 2\Delta x^T H^T R^{-1} z \\
 &\quad + 2\Delta x^T (H^T R^{-1} H + \lambda I) \hat{x} + \Delta x^T (H^T R^{-1} H + \lambda I) \Delta x - 2\lambda \Delta x^T c \\
 &\quad - 2\lambda \hat{x}^T c + \lambda c^T c \\
 &= z^T R^{-1} z - \hat{x}^T H^T R^{-1} z + \hat{x}^T \left[(H^T R^{-1} H + \lambda I) \hat{x} - H^T R^{-1} z - \lambda c \right] \\
 &\quad + 2\Delta x^T \left[(H^T R^{-1} H + \lambda I) \hat{x} - H^T R^{-1} z - \lambda c \right] \\
 &\quad + \Delta x^T (H^T R^{-1} H + \lambda I) \Delta x - \lambda \hat{x}^T c + \lambda c^T c
 \end{aligned}$$

The third and fourth terms vanish. We next substitute for \hat{x}^T in the second term and obtain

$$J_m = z^T R^{-1} z - (H^T R^{-1} z + \lambda c)^T (H^T R^{-1} H + \lambda I)^{-1} H^T R^{-1} z \\ + \Delta x^T (H^T R^{-1} H + \lambda I) \Delta x - \lambda \hat{x}^T c + \lambda c^T c$$

Expanding the second term gives

$$J_m = z^T \left[R^{-1} - R^{-1} H (H^T R^{-1} H + \lambda I)^{-1} H^T R^{-1} \right] z - \lambda c^T (H^T R^{-1} H + \lambda I)^{-1} H^T R^{-1} z \\ + \Delta x^T (H^T R^{-1} H + \lambda I) \Delta x - \lambda \hat{x}^T c + \lambda c^T c$$

We next add and subtract the term $\lambda^2 c^T (H^T R^{-1} H + \lambda I)^{-1} c - \lambda c^T \hat{x}$ and obtain for J_m

$$J_m = z^T \left[R^{-1} - R^{-1} H (H^T R^{-1} H + \lambda I)^{-1} H^T R^{-1} \right] z \\ - \lambda c^T \left[(H^T R^{-1} H + \lambda I)^{-1} H^T R^{-1} z + \lambda (H^T R^{-1} H + \lambda I)^{-1} c - \hat{x} \right] \\ - \lambda^2 c^T (H^T R^{-1} H + \lambda I)^{-1} c + \lambda c^T \hat{x} - \lambda \hat{x}^T c + \lambda c^T c \\ + \Delta x^T (H^T R^{-1} H + \lambda I) \Delta x$$

The second term vanishes and two terms cancel leaving

$$\begin{aligned}
 J_m &= z^T \left[R^{-1} - R^{-1} H (H^T R^{-1} H + \lambda I)^{-1} H^T R^{-1} \right] z \\
 &\quad + \lambda c^T \left[I - \lambda (H^T R^{-1} H + \lambda I)^{-1} \right] c \\
 &\quad + \Delta x^T (H^T R^{-1} H + \lambda I) \Delta x
 \end{aligned}$$

If λ is set equal to zero this reduces to the case considered in Section III. As in Section III, we can now write the recurrence relation for $f_{m+1}[z(m+1), c, \lambda]$ as

$$\begin{aligned}
 f_{m+1}[z(m+1), c, \lambda] &= \text{Max}_{\Delta x} \left\{ \left[z_{m+1} - H_{m+1} [(\hat{x})_{m-n} + \Delta x] \right]^T R_{m+1}^{-1} \left[z_{m+1} \right. \right. \\
 &\quad \left. \left. - H_{m+1} [(\hat{x})_{m-n} + \Delta x] \right] + \Delta x^T [H^T(m) R^{-1}(m) H(m) + \lambda I] \Delta x \right. \\
 &\quad \left. + f_m[z(m), c, \lambda] \right\}
 \end{aligned}$$

where

$$\begin{aligned}
 f_n[z(n), c, \lambda] &= z^T(n) \left\{ R^{-1}(n) - R^{-1}(n) H(n) \left[H^T(n) R^{-1}(n) H(n) \right. \right. \\
 &\quad \left. \left. + \lambda I \right]^{-1} H^T(n) R^{-1}(n) \right\} z(n) + \lambda c^T \left\{ I - \lambda \left[H^T(n) R^{-1}(n) H(n) \right. \right. \\
 &\quad \left. \left. + \lambda I \right]^{-1} \right\} c
 \end{aligned}$$

$m = n, n+1, \dots$

Comparing this recurrence relation with that in Section III for $f_{m+1} [z(m+1)]$ shows that the terms involving vector Δx in the two expressions differ only by having $H^T(m) R^{-1}(m) H(m)$ replaced by $H^T(m) R^{-1}(m) H(m) + \lambda I$. If we make a similar substitution in the equation for $(\Delta \hat{x})_{m-n}$ we obtain

$$(\Delta \hat{x})_{m-n} = \left[H^T(m) R^{-1}(m) H(m) + H_{m+1}^T R_{m+1}^{-1} H_{m+1} + \lambda I \right]^{-1} H_{m+1}^T R_{m+1} \left[z_{m+1} - H_{m+1} (\hat{x})_{m-n} \right]$$

Define

$$P_m^{-1} = H^T(m) R^{-1}(m) H(m) + \lambda I$$

$$P_{m+1}^{-1} = P_m^{-1} + H_{m+1}^T R_{m+1}^{-1} H_{m+1}$$

From the lemma in Ref. 1, we have, as in Section III

$$P_{m+1} = P_m - P_m H_{m+1}^T (H_{m+1} P_m H_{m+1}^T + R_{m+1})^{-1} H_{m+1} P_m$$

We can thus use the final set of recurrence relations given in Section III for ill-conditioned systems. The difference for ill-conditioned systems concerns the initial matrix P_0 , which is given by

$$P_0 = H^T(n) R^{-1}(n) H(n) + \lambda I$$

and the initial estimate $(\hat{x})_0$, which is obtained by the method of successive approximations as described in Ref. 4 using the first n observations.

V. THE ORBIT DETERMINATION PROBLEM

The determination of the definitive orbit of a celestial object usually involves a differential correction process which relates observation residuals to corrections of a set of two-body orbital parameters.* This problem can be quite easily treated as a Kalman filtering problem, since the observational data are usually large in number and the sources can be separated in space and time. This application was originally proposed by Swerling.⁽⁶⁾

As an example, consider the differential correction process discussed in Ref. 7. The residuals of a sequence of azimuth and elevation angle observations are to be related to the corrections to a set of six two-body orbital parameters. These corrections can be represented as an orbital parameter correction vector

$$\Delta p = [\Delta a/a, \Delta(e \cos E_0), \Delta(e \sin E_0), \Delta \tilde{u}_0, \Delta \tilde{v}_0, \Delta \tilde{w}_0]^T$$

where⁽⁸⁾

a is the semi-major axis

e is the eccentricity

E_0 is the value of the eccentric anomaly at $t = t_0$

$\Delta \tilde{u}_0, \Delta \tilde{v}_0, \Delta \tilde{w}_0$ are small rotations about a set of unit vectors $\vec{U}_0, \vec{V}_0, \vec{W}$ which define the orientation of the orbit in space.

The small rotations $\Delta \tilde{u}_0, \Delta \tilde{v}_0$, and $\Delta \tilde{w}_0$ can be related to small changes in the unit vectors $\Delta \vec{U}_0, \Delta \vec{V}_0, \Delta \vec{W}$. The vector Δp corresponds to vector x in Sections II and III.

* See Chapter 11 of Ref. 5 for a discussion of this problem.

The true values of the observation residuals are represented as the $m \times 1$ vector

$$\Delta q = [\Delta q_1, \Delta q_2, \dots, \Delta q_m]^T$$

where

$$\left. \begin{aligned} \Delta q_i &= -\rho_i \Delta A_i \cos h_i \\ \Delta q_{i+1} &= \rho_i \Delta h_i \end{aligned} \right\} \quad i = 1, 3, \dots, m-1$$

The quantities ΔA_i and Δh_i are the true values of the observation residuals for the azimuth and elevation angles.

Within the limits of linearity the vectors Δq and Δp are related by the relation

$$\Delta q = C \Delta p$$

where C is an $m \times 6$ matrix whose elements are functions of time and the elements of the preliminary two-body orbit being corrected. The explicit expressions giving the elements of matrix C are rather involved. Since they are derived in detail in Ref. 7, they will not be given here.

If we assume that the observations are corrupted by white gaussian noise of zero mean, then the above vector matrix equation becomes

$$v = z - C \Delta p$$

where the observation vector in Sections II and III is given by

$$z = v + \Delta q$$

and

$$v = [v_1, \dots, v_m]^T$$

The components of observation vector z are

$$\left. \begin{aligned} z_i &= -\rho_i (A_{oi} - A_{ci}) \cos h_i \\ z_{i+1} &= \rho_i (h_{oi} - h_{ci}) \end{aligned} \right\} \quad i = 1, 3, \dots, m-1$$

where A_{oi} , h_{oi} are the observed and A_{ci} , h_{ci} are the computed values of azimuth and elevation angles at times t_i . Angles A_{ci} and h_{ci} are based on the values of the orbital parameters of the preliminary orbit being corrected, as are the factors ρ_i and $\cos h_i$.

By associating $\Delta \hat{p}_k$ with \hat{x}_k , $\delta \hat{p}_k$ with $\Delta \hat{x}_k$, and matrix C with matrix H the recurrence relations in Section III become

$$\left. \begin{aligned} \delta \hat{p}_k &= P_{k+1} C_{k+1}^T R_{k+1}^{-1} (z_{k+1} - C_{k+1} \Delta \hat{p}_k) \\ \Delta \hat{p}_{k+1} &= \Delta \hat{p}_k + \delta \hat{p}_k \\ P_{k+1} &= P_k - P_k C_{k+1}^T (C_{k+1} P_k C_{k+1}^T + R_{k+1})^{-1} C_{k+1} P_k \end{aligned} \right\} \quad k = 1, 2, \dots$$

where the subscript k refers to the k -th estimate.

If we assume that each observation results in a pair of residuals in azimuth and elevation, then matrix C_k is 2×6 , matrix P_k is 6×6 , and matrix R_k is 2×2 . Vector z_k will be 2×1 .

If we take the first three pairs of observations we can form the equation

$$z = \begin{bmatrix} z_1 \\ - \\ z_2 \\ - \\ z_3 \end{bmatrix} = \begin{bmatrix} C_1 \\ - \\ C_2 \\ - \\ C_3 \end{bmatrix} \Delta p = C \Delta p$$

where

$$z_i = \left[-\rho_i (A_{oi} - A_{ci}) \cos h_i, \rho_i (h_{oi} - h_{ci}) \right]^T, i = 1, 2, 3$$

We can compute the initial estimate of Δp as*

$$\Delta \hat{p}_1 = C^{-1} z$$

Azimuth and elevation angle observations are independent. Assume they have equal variances σ^2 which are constant for all observations.

Then

$$R_1 = R_2 = R_3 = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix} = \sigma^2 I_2$$

Let

$$R = \begin{bmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & R_3 \end{bmatrix} = \sigma^2 I_6$$

Then since P_1 is the covariance matrix for $\Delta \hat{p}_1$ it can be computed from

$$P_1 = \sigma^2 (C^T C)^{-1}$$

It is also possible to let P_1 be a diagonal matrix whose elements are estimated on the basis of experience, and to let $\Delta \hat{p}_1$ be the null vector.

*If the matrix C is singular then the observations are not independent and the whole process breaks down. The process discussed in Section IV can be used if the observations are ill-conditioned.

VI. THE FEEDBACK CONTROL PROBLEM

THE ORBIT TRANSFER PROBLEM AS AN OPTIMAL REGULATOR PROBLEM

An orbit transfer process can be represented as a feedback control process and treated as an optimal regulator problem. Consider that the space vehicle is in an orbit defined by p_0 at time t_0 and it is desired to transfer it to an orbit defined by p_τ at time $t_0 + \tau$. The symbols p_0 and p_τ represent vectors having as components the sets of values of the two-body orbital parameters describing the osculating orbits to the vehicle's trajectories at t_0 and $t_0 + \tau$. We define an orbital correction vector at time t as

$$\Delta p(t) = p_\tau - p(t)$$

where $p(t)$ is the vector representing the values of the osculating orbital parameters at time t . The object of the orbit transfer process is then to drive the orbital parameter correction vector to zero at time $t_0 + \tau$. By some optimization process the optimal control vector can be expressed as a function of the orbital parameter correction vector $\Delta p(t)$. This closes a feedback loop and expresses the orbit transfer problem as an optimal regulator problem. It also implies the necessity for computing the components of $\Delta p(t)$ as functions of certain observed quantities, since the components of $\Delta p(t)$ are not directly observable.*

The orbit transfer problem can be formulated in phase space. In this case vectors p_0 and p_τ are replaced by s and s_r . The components of vectors s and s_r are the rectangular components of position and velocity in the initial and terminal trajectories. A relative system

* Assuming $\Delta p(t)$ is inaccurately known.

state vector can then be defined as

$$\begin{aligned}\Delta s(t) &= s_r(t) - s(t) \\ &= [\Delta x(t), \Delta \dot{x}(t), \Delta y(t), \Delta \dot{y}(t), \Delta z(t), \Delta \dot{z}(t)]^T\end{aligned}$$

In phase space the values of the components of the system state vectors s and s_r change with time, with or without thrust being applied to the vehicle. The object of the orbit transfer problem is then to make state vector s coincide with vector s_r at time $t_o + \tau$.

The vector $s_r(t)$ can represent the motion of a hypothetical point mass moving along the desired terminal trajectory of the orbit transfer process, or it can represent the motion of an actual space vehicle that is to be intercepted or rendezvoused with. In the intercept case it may be desired only to drive the relative position components $\Delta x(t)$, $\Delta y(t)$, and $\Delta z(t)$ to zero at time $t_o + \tau$.

As the components of $\Delta s(t)$ are usually not directly observable, it is necessary to compute them from observable quantities in order that the optimal control process can be carried out.

Actually, in phase space, the orbit transfer is more like a servo problem than a regulator problem since $\Delta s(t)$ refers to a moving origin. However, under certain circumstances it can be treated approximately as a regulator problem by neglecting certain small terms or defining $\Delta s(t)$ as $\Delta s(t) = s_r(t_o + \tau) - s(t)$.

In the remainder of this section the application of Kalman filtering theory to the estimation of orbital parameter correction vectors $\Delta p(t)$ and relative system state vectors $\Delta s(t)$ will be discussed.

THE ESTIMATION OF ORBITAL PARAMETER CORRECTION VECTORS

In the discussion to follow it is assumed that the durations of time involved are short enough so that the space vehicle essentially follows a two-body orbit while no thrust is being applied, i.e., the effect of perturbations from natural causes is assumed negligible. Under these conditions the motion of the body is defined by a set of six constant orbital parameters. The rectangular position and velocity of the space vehicle can be determined at any particular time from these orbital parameters by applying suitable transformation equations. However, when thrust is applied to the space vehicle the values of these orbital parameters change continuously with time. At each instant of time the resulting set of orbital parameters defines an osculating conic. This conic represents the path in space along which the vehicle would move if the thrust were to vanish. An orbit transfer can thus be considered as the time behavior of a two-body orbit that is continuously deformed in size, shape, and orientation in space from some initial configuration to some terminal configuration. Because of the nonlinearity of the set of differential equations defining the time behavior of the orbital parameters, the estimation problem will be treated as a discrete process and linear approximations used.

The orbit determination problem involves obtaining the estimated values of a set of six constant parameters from a sequence of observations. The feedback control problem involves determining the estimated values of a set of six time-varying orbital parameters at discrete instants of time during the orbit transfer process. Thus, some relation is necessary for determining how the orbital parameters change their

values with time.

A piece-wise linear solution to an optimal control problem formulated in terms of orbital parameters and obtained by dynamic programming is given in Ref. 9. The linear transformation relating the orbital parameter correction vector at time t_k to that at time t_{k+1} is given by

$$\Delta p_{k+1} = \Delta p_k - A_k \Delta V_k, \quad k = 0, 1, \dots, N-1$$

where A_k is a 6×3 matrix whose elements are evaluated from the components of the osculating orbital parameter vector at t_k . The vector ΔV_k is the incremental change in the space vehicle's velocity vector from t_k to t_{k+1} due to the application of thrust. From Ref. 9

$$\Delta V_k = (A_k^T Q_{k+1} A_k + \lambda I)^{-1} A_k^T Q_{k+1} \Delta p_k, \quad k = 0, 1, \dots, N-1$$

where Q_{k+1} , a 6×6 matrix, is determined from

$$Q_k = Q_{k+1} - Q_{k+1} A_k (A_k^T Q_{k+1} A_k + I)^{-1} A_k^T Q_{k+1}$$

$$Q_N = I$$

Substituting from ΔV_k in the transformation equation yields

$$\Delta p_{k+1} = \left[I - A_k (A_k^T Q_{k+1} A_k + \lambda I)^{-1} A_k^T Q_{k+1} \right] \Delta p_k$$

Multiplying both sides by matrix Q_{k+1}^{-1} and solving for Δp_k gives

$$\Delta p_k = Q_k^{-1} Q_{k+1} \Delta p_{k+1}$$

This relation defines how the orbit parameter correction vector changes from one stage of the process to the next during an optimal orbit transfer.

It will be assumed that at each time t_k an observation $z(t_k)$ is obtained, an orbital parameter correction vector estimate $\Delta p(t_k)$ will be computed (Fig. 1). The optimal control process will utilize this estimate to compute a change in the vehicle's velocity vector. The orbital parameter correction vector and the true values of the observations at t_k are related by

$$\Delta q(t_k) = C(p_\tau; t_k) \Delta p(t_k)$$

Matrix $C(p_\tau; t_k)$ is the $m \times 6$ matrix discussed in Section IV except that its elements are evaluated in terms of the components of p_τ . This means that the linear approximations involved in deriving the elements of matrix C become more exact as the termination of the process is approached.

Associating $\Delta q(t_{k+1})$ with y_{m+1} , $\delta \hat{p}(t_{k+1})$ with $\Delta \hat{x}_m$, $\Delta \hat{p}(t_{k+1})$ with \hat{x}_{m+1} , and $C(p_\tau; t_{k+1})$ with H_{m+1} , we can rewrite the recurrence relations with a necessary slight modification as

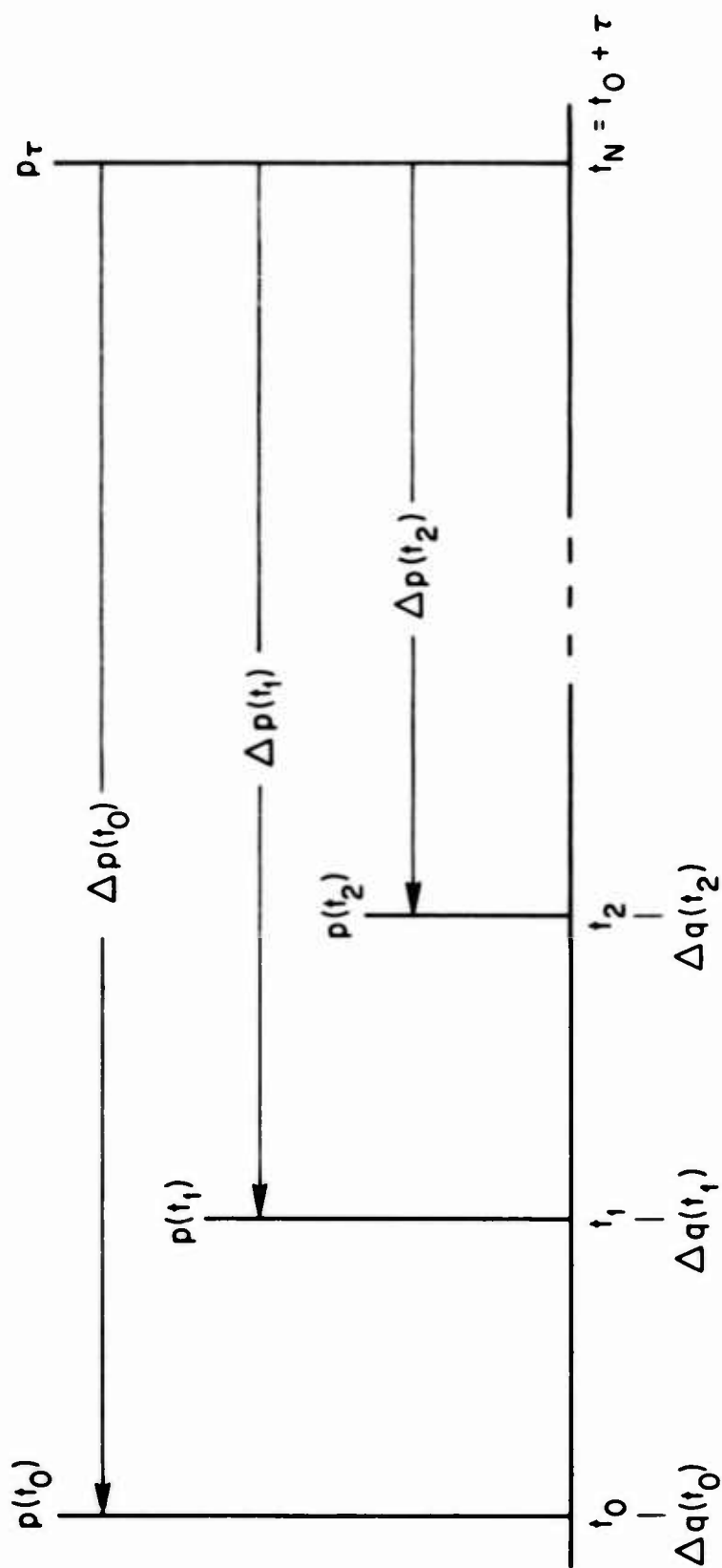
$$\delta \hat{p}_{k+1} = P_{k+1} C_{k+1}^T R_{k+1}^{-1} \left[z(t_{k+1}) - C_{k+1} Q_{k+1}^{-1} Q_k \Delta \hat{p}(t_k) \right]$$

$$\Delta \hat{p}(t_{k+1}) = Q_{k+1}^{-1} Q_k \Delta \hat{p}(t_k) + \delta \hat{p}(t_{k+1})$$

$$P_{k+1} = P_k - P_k C_{k+1}^T (C_{k+1}^T P_k C_{k+1} + R_{k+1})^{-1} C_{k+1} P_k$$

where

$$C_{k+1} = C(p_\tau; t_{k+1})$$



$$\Delta q(t_k) = C(p_\tau; t_k) \Delta p(t_k)$$

Fig. 1—Relations among orbital parameter vectors, orbital parameter correction vectors, and observations

The slight modification mentioned above involves the extrapolation of $\Delta \hat{p}(t_k)$ ahead to time t_{k+1} by using the Q matrices. The sequence of Q matrices is generated in the process of solving the optimal control problem.

THE ESTIMATION OF RELATIVE SYSTEM STATE VECTORS

The linearized time-varying differential equation for the relative system state vector is given⁽¹⁰⁾ by

$$\frac{d\Delta s}{dt} = A(t) \Delta s - [A(t) - A_r] s_r(t) - Bu(t), \quad \Delta s(t_0) = \Delta c$$

where $A(t)$ is a time-varying matrix, A_r is a constant matrix for the target in a circular orbit, B is a constant matrix, and $u(t)$ is the control vector. For a quadratic system performance index the optimal control vector will be a linear function of the relative system state vector

$$u(t) = \frac{1}{\lambda} B^T \left[\alpha(\tau + t_0 - t) \Delta s(t) + \beta(\tau + t_0 - t) s_r(t) \right]$$

Substituting this expression for $u(t)$ in the differential equation above yields

$$\frac{d\Delta s}{dt} = F(\tau + t_0 - t) \Delta s + G(\tau + t_0 - t) s_r(t)$$

where

$$F(\tau + t_0 - t) = A(t) - \frac{1}{\lambda} B B^T \alpha(\tau + t_0 - t)$$

$$G(\tau + t_0 - t) = A_r - A(t) - \frac{1}{\lambda} B B^T \beta(\tau + t_0 - t)$$

The solution of this time-varying differential equation is given by

$$\begin{aligned}\Delta s(t) &= \Phi(t, t_0) \Delta s(t_0) + \int_{t_0}^t \Phi(t, \xi) G(\xi) s_r(\xi) d\xi \\ &= \Phi(t, t_0) \Delta s(t_0) + g(t, t_0)\end{aligned}$$

where matrix $\Phi(t, t_0)$ satisfies the matrix differential equation

$$\frac{d\Phi}{dt} = F(\tau + t_0 - t) \Phi, \quad \Phi(t_0, t_0) = I$$

This solution defines the time behavior of the relative state vector as the vehicle moves along a trajectory determined by the optimal control vector.

The relative system state variables are expressed with respect to a non-rotating coordinate system and are not directly observable. Their estimated values must therefore be based on physical quantities that can be observed. There is obviously more than one way of handling this problem. The method to be discussed consists of two separate computational processes.

- Observations of relative range, relative range rate, two angles, and two angular rates associated with the line of sight from $s(t)$ to $s_r(t)$ are made at discrete instants of time. The corresponding values of the state variables are computed by suitable transformation equations.
- The sequence of sets of computed values of the state variables is operated on by a Kalman filter to optimally estimate values of the state variables updated in time from the observation times.

Assume that $\Delta \tilde{s}(t_k)$, $k = 1, 2, \dots, m$ represents the computed relative system state vector based on observations at t_k , $k = 1, 2, \dots, m$. The problem is to estimate the $\Delta s(T)$ where $T > t_k$, $k = 1, 2, \dots, m$.

The relative system state vector at $t = T$ is related to the relative system state vector at t_k by the solution of the differential equation

$$\begin{aligned} \Delta s(T) &= \Phi(T, t_k) \Delta s(t_k) + \int_{t_k}^T \Phi(T, \xi) G(\xi) s_r(\xi) d\xi \\ &= \Phi(T, t_k) \Delta s(t_k) + g(T, t_k) \end{aligned}$$

or

$$\Delta s(t_k) = \Phi^{-1}(T, t_k) [\Delta s(T) - g(T, t_k)]$$

By making the proper associations of vectors and matrices we can rewrite the recurrence relations in Section III as

$$\begin{aligned} [\hat{\Delta s}(T)]_k &= P_{k+1} \Phi^T(t_{k+1}, T) R_s^{-1}(t_{k+1}) [\Delta \tilde{s}(t_{k+1}) \\ &\quad - \Phi(t_{k+1}, T) \{[\hat{\Delta s}(T)]_k - g(T, t_{k+1})\}] \end{aligned}$$

$$[\hat{\Delta s}(T)]_{k+1} = [\hat{\Delta s}(T)]_k + [\delta \hat{s}(T)]_k$$

$$P_{k+1} = P_k - P_k \Phi^T(t_{k+1}, T) [\Phi(t_{k+1}, T) P_k \Phi^T(t_{k+1}, T)$$

$$+ R_s(t_{k+1})]^{-1} \Phi(t_{k+1}, T) P_k$$

where

$$\phi(t_{k+1}, T) = \phi^{-1}(T, t_{k+1})$$

$$g(T, t_k) = \int_{t_k}^T \phi(T, \xi) G(\xi) s_r(\xi) d\xi$$

$$k = 0, 1, \dots, m-1$$

There will be some uncertainty associated with $g(T, t_k)$ due to uncertainty in knowledge of the target's orbit. However, the matrix $G(\xi)$ will have elements that are either relatively small or zero, and since $g(T, t_k)$ is in the nature of a correction the effects of uncertainties associated with $g(T, t_k)$ will be ignored.

The initial quantities required by the recurrence relations are $[\Delta \hat{s}(T)]_0$, P_0 , and $R_s(t_1)$. The initial estimate of $[\Delta \hat{s}(T)]_0$ can be computed from

$$[\Delta \hat{s}(T)]_0 = \phi(T, t_1) \Delta \tilde{s}(t_1)$$

The matrix P_0 can be taken as a diagonal matrix whose non-zero elements are the variances associated with the initial values of the relative system state variables.

If the matrix $R_q(t_k)$ is the covariance matrix associated with the set of observations taken at time t_k and the matrix $M(t_k)$ is the Jacobian matrix associated with the equations transforming from the observed quantities to the computed relative system state variables, then when

the observation uncertainties are sufficiently small

$$R_s(t_k) = M(t_k) R_q(t_k) M^T(t_k)$$

where $R_s(t_k)$ is the covariance matrix for the relative system state variables at t_k .

The process just described estimates the values of the relative system state variables at time $T > t_k$, $k = 1, 2, \dots, m$ based on m sets of observations. The feedback control process requires that the relative system state variables be estimated either continuously or at a discrete number of times throughout the orbit transfer process. Since the computational processes involved will usually be performed by a digital computer, both cases are actually discrete, the difference being due to the frequency with which $\Delta\hat{s}(t)$ is estimated.

The period of time during which observational data are processed to obtain the estimate $\Delta\hat{s}(T)$, i.e., $0 \leq t \leq T$ will be referred to as an estimation interval. It is assumed that the orbit transfer process consists of N estimation intervals, i.e., $\Delta s(t)$ is estimated at times $T, 2T, \dots, (N-1)T$ with m sets of observations being made during each estimation interval. The process involved in estimating $\Delta\hat{s}(T)$ is then repeated during each estimation interval, and it remains to consider how the various estimation intervals are linked together so that all preceding observational data are used for each successive estimate.

The initial estimate of $\Delta\hat{s}(2T)$ can be computed from

$$[\Delta\hat{s}(2T)]_0 = \phi(2T, T) [\Delta\hat{s}(T)]_{m-1} + g(2T, T)$$

where

$$g(2T, T) = \int_T^{2T} \Phi(2T, \xi) G(\xi) s_r(\xi) d\xi$$

The initial P matrix for the second estimation interval is computed from

$$P_m = P_{m-1} - P_{m-1} [P_{m-1} + R_m]^{-1} P_{m-1}$$

since

$$\Phi(t_m, T) = \Phi(T, T) = I$$

The computation of $R_s(t)$ depends only on the time of the associated set of observations, and the equation given above is used for all estimation intervals.*

Imbedding the one-stage estimation process in the N-stage estimation process provides estimates of the relative system state variables at (N-1) instants of time during the orbit transfer. When a continuous control process is involved, the digital computation process used to compute $\Delta \hat{s}(t)$ is based on a simple linear extrapolation during each estimation interval; i.e.,

$$\Delta \hat{s}(t) = \Phi(t, nT) \Delta \hat{s}(nT) + g(t, nT), \quad n = 1, 2, \dots, N-1$$

where

$$g(t, nT) = \int_{nT}^t \Phi(t, \xi) G(\xi) s_r(\xi) d\xi$$

* Since the matrix $M(t)$ becomes singular as $s(t)$ approaches $s_r(t)$ the computation process for $R_s(t)$ may have to be modified during the last few estimation intervals.

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